NONCOMMUTATIVE SPACES WITH TWISTED SYMMETRIES AND SECOND QUANTIZATION

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Abstract

In a minimalistic view, the use of noncommutative coordinates can be seen just as a way to better express non-local interactions of a special kind: 1-particle solutions (wavefunctions) of the equation of motion in the presence of an external field may look simpler as functions of noncommutative coordinates. It turns out that also the wave-mechanical description of a system of n such bosons/fermions and its second quantization is simplified if we translate them in terms of their deformed counterparts. The latter are obtained by a general twist-induced $\star$-deformation procedure which deforms in a coordinated way not just the spacetime algebra, but the larger algebra generated by any number n of copies of the spacetime coordinates and by the particle creation and annihilation operators. On the deformed algebra the action of the original spacetime transformations looks twisted.

In a non-conservative view, we thus obtain a twisted covariant framework for QFT on the corresponding noncommutative spacetime consistent with quantum mechanical axioms and Bose-Fermi statistics. One distinguishing feature is that the field commutation relations remain of the type "field (anti)commutator=a distribution". We illustrate the results by choosing as examples interacting non-relativistic and free relativistic QFT on Moyal space(time)s.

1 Introduction

Second Quantization played a crucial role in the foundation of Quantum Field Theory (QFT) as a bottom-up approach from the wave-mechanical description of a system of n identical quantum particles. The nonrelativistic field operator of a spinless particle in $\mathbb{R}^3$ (in the Schrödinger picture) and its hermitean conjugate were introduced by

$$\varphi(\mathrm{x}) := \varphi_i(\mathrm{x})a^i, \qquad \varphi^*(\mathrm{x}) = \varphi_i^*(\mathrm{x})a_i^+, \qquad \text{(infinite sum over } i), \tag{1}$$

where $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of the 1-particle Hilbert space and φ_i, a_i^+, a^i the wavefunction, creation, annihilation operators associated to e_i. Here we summarize how to perform [17] second quantization on a non-commutative space using a *twist* [11] to deform in a coordinated way space(time), its symmetries and all objects transforming under space(time) transformations. This is an alternative to the other approaches to QFT adopted so far, namely path-integral approaches [12], or operator approaches where the field commutation relations are fixed by other prescriptions, e.g. adapted canonical

quantization (see e.g. [10, 4]) or Wightman axioms (see e.g. [2, 8, 15]); moreover, as in [18, 7, 8, 1, 15], it aims at recovering all the undeformed spacetime symmetries in terms of noncocommutative Hopf algebras .

A rather general way to deform an algebra $\mathcal{A}$ (over $\mathbb{C}$, say) into a new one $\mathcal{A}_\star$ is by deformation quantization [5]. Calling λ the deformation parameter, this means that the two have the same vector space over $\mathbb{C}[[\lambda]]$, $V(\mathcal{A}_\star) = V(\mathcal{A})[[\lambda]]$, but the product $\star$ of $\mathcal{A}_\star$ is a deformation of the product $\cdot$ of $\mathcal{A}$. On the algebra $\mathcal{X}$ of smooth functions on a manifold X, and on the algebra $\mathcal{D} \supset \mathcal{X}$ of differential operators on $\mathcal{X}$, $f \star h$ can be defined applying to $f \otimes h$ first a suitable bi-pseudodifferential operator $\overline{\mathcal{F}}$ (depending on the deformation parameter λ and reducing to the identity when $\lambda = 0$) and then the pointwise multiplication $\cdot$. The simplest example is probably provided by the Moyal $\star$-product on $X = \mathbb{R}^m$:

$$\begin{aligned} &a(x) \star b(x) := a(x) \exp\left[\tfrac{i}{2} \overleftarrow{\partial_h}\, \lambda \vartheta^{hk} \overrightarrow{\partial_k}\right] b(x) = \cdot \left[\overline{\mathcal{F}}(\triangleright \otimes \triangleright)(a \otimes b)\right], \\ &\overline{\mathcal{F}} := \exp\left(-\tfrac{i}{2} \theta^{hk} P_h \otimes P_k\right), \qquad\qquad \theta^{hk} := \lambda \vartheta^{hk}, \end{aligned} \tag{2}$$

where P_h are the generators of translations (on $\mathcal{X}$ P_h can be identified with $-i\partial_h := i\partial/\partial x^h$), and ϑ^{hk} is a fixed real antisymmetric matrix (this is well-defined for polynomials or other infinitely differentiable functions a, b such that the λ-power series $(2)_1$ converges; an alternative definition (36) in terms of Fourier transforms makes sense on much larger domains).

If one replaces all $\cdot$ by $\star$'s in an equation of motion, e.g. in the Schrödinger equation on $X = \mathbb{R}^3$ of a particle with electrical charge q

$$\mathsf{H}_\star^{(1)} \psi(\mathrm{x}) = i\hbar \partial_t \psi(\mathrm{x}), \qquad \mathsf{H}_\star^{(1)} := \left[\tfrac{-\hbar^2}{2m} D^a \star D_a + V\right] \star, \qquad D_a = \partial_a + iqA_a, \tag{3}$$

one obtains a pseudodifferential equation and therefore introduces a moderate (very special) amount of non-locality in the interactions. In the case of the Moyal $\star$-product on $X = \mathbb{R}^3$ this becomes

$$\tfrac{-\hbar^2}{2m} \partial_h \partial^h \psi(\mathrm{x}) + V(\mathrm{x}) \exp\left[\tfrac{i}{2} \overleftarrow{\partial_h}\, \theta^{hk} \overrightarrow{\partial_k}\right] \psi(\mathrm{x}) = E\psi(\mathrm{x})$$

where we have chosen for simplicity $A = 0$. The kinetic part is undeformed, non-locality is concentrated only in the interactions. $\mathcal{X}_\star, \mathcal{X}$ have the same Poincaré-Birkhoff-Witt series, i.e. the subspaces of $\star$-polynomials and $\cdot$-polynomials of any fixed degree in x^h coincide. The algebras $\mathcal{X}, \mathcal{X}_\star$ can be defined through the same generators (i.e. coordinates x^h and $\mathbf{1}$) and different (but related) relations. One can define a linear map $\wedge : f \in \mathcal{X}[[\lambda]] \to \hat{f} \in \mathcal{X}_\star$ by the requirement that it reduces to the identity on the vector space $V(\mathcal{X}_\star) = V(\mathcal{X})[[\lambda]]$: $\hat{f}(\mathrm{x}\star) = f(\mathrm{x})$. One finds

$$\begin{aligned} &\wedge(x^h) = x^h \\ &\wedge(x^h x^k) = x^h \star x^k - \tfrac{i}{2}\theta^{hk} \qquad \Rightarrow \qquad [x^h \overset{\star}{,} x^k] = \mathbf{1} i \theta^{hk} \\ &\quad \dots \end{aligned} \tag{4}$$

In other words, by $\wedge$ one expresses functions of x^h as functions of $\mathrm{x}^h\star$. Similarly one proceeds with $\mathcal{D}, \mathcal{D}_\star$. $\wedge$ transforms (3) into a $\star$-differential equation of second order (i.e. of second degree in $\partial_h\star$), what may critically simplify the study of the equation:

$$\tfrac{-\hbar^2}{2m} \partial_h \star \partial^h \star \hat{\psi}(\mathrm{x}\star) + \hat{V}(\mathrm{x}\star) \star \hat{\psi}(\mathrm{x}\star) = E\hat{\psi}(\mathrm{x}\star),$$

How does a $\star$-product transform under a transformation of the Euclidean group G, the symmetry group of $\mathbb{R}^3$, or equivalently under the action of the Universal Enveloping Algebra (UEA) $U\mathbf{g}$ of the Lie algebra $\mathbf{g}$ of G? According to the coproduct of a noncocommutative Hopf algebra obtained deforming $U\mathbf{g}$ by the twist $\mathcal{F}$ inverse of $(2)_2$ (section **??**) .
Actually, in section **??** we are going to present a procedure which applies to a large class of twist-induced $\star$-deformations of $X=\mathbb{R}^m$ (or of symmetric submanifolds X of $\mathbb{R}^m$) and of the spacetime symmetry covariance group G of the quantum theories on X, leading to results generalizing the ones sketched above for the Moyal deformations.
What about multiparticle systems with a non-local interaction of the above kind? Their description will be simplified if we use generators $x_j^h\star$, $\partial_h^j\star$, $j=1,...,n$. However we respect Bose/Fermi statistics, i.e. the rule to compute the number of allowed states of n identical bosons/fermions. Second quantization will be simplified if we also use generators $a^i\star$, $a_i^+\star$, $i\in\mathbb{N}$. In general, we shall expand all products $\cdot$'s in terms of $\star$-products

$$f\star g = f\cdot g+\lambda(\overline{\mathcal{F}}^\alpha\triangleright f)\cdot(\overline{\mathcal{F}}_\alpha\triangleright g)+O(\lambda^2) \qquad \Rightarrow \qquad f\cdot g = f\star g-\lambda(\overline{\mathcal{F}}^\alpha\triangleright f)\star(\overline{\mathcal{F}}_\alpha\triangleright g)+O_\star(\lambda^2)$$

in *all commutative notions* [wavefuncts ψ, diff. operators (Hamiltonian, etc), $a^i, a_i^\dagger$,..., action of H, second quantization itself] to introduce *their noncommutative analogs.* Then forgetting the $\cdot$'s we end up with a "noncommutative way" to look at QFT, or a *noncommutative space(time) and a (formal) closed framework for covariant QFT on it.*

A similar strategy has been used by J. Wess & collaborators [22, 3] to formulate noncommutative diffeomorphisms and related notions (metric, connections, tensors etc).

Alternatively, if one prefers a minimalistic view one can keep a *commutative* spacetime and use $\star$'s only to introduce peculiar non-local interactions; then the use of noncommutative coordinates may be seen just as a help to solve the dynamics.

In section **??** we describe the twist-induced deformation of a cocommutative Hopf $*$-algebra and of its module $*$-algebras, in particular the algebras of functions and of differential operators on symmetric submanifolds X of $\mathbb{R}^m$ and the Heisenberg/Clifford algebra associated to bosons/fermions on X. In section **??** we use these tools to deform the (non-relativistic) wave-mechanical formulation of a system of bosons/fermions on X and the Second Quantization of the latter; we also study a charged particle in a constant magnetic field $\mathbf{B}$ on the Moyal deformation of $\mathbb{R}^3$ as an example of a model where the use of noncommutative coordinates helps solving the dynamics (3). In section **??** we extend the Second Quantization procedure to relativistic free fields on a deformed Minkowski spacetime covariant under the associated deformed Poincaré Group, devoting attention in particular to the Moyal-Minkowski spaces and the corresponding twisted Poincaré Hopf algebra $\widehat{U\mathcal{P}}$ [7, 22, 18].
We denote as $V(\mathcal{A})$ the vector space underlying an algebra $\mathcal{A}$. We stick to linear spaces and algebras over $\mathbb{C}$ or the ring $\mathbb{C}[[\lambda]]$ of formal power series in λ with coefficients in $\mathbb{C}$; then tensor products are to be understood as completed in the λ-adic topology. We shall often change notation: $\mathcal{X}_\star\rightsquigarrow\hat{\mathcal{X}}$, $\mathcal{D}_\star\rightsquigarrow\hat{\mathcal{D}}$, $x_j^h\star\rightsquigarrow\hat{x}_j^h$, $\partial_h^j\star\rightsquigarrow\hat{\partial}_h^j$, $a_i^+\star\rightsquigarrow\hat{a}_i^+$, etc. For instance, the previous Schrödinger equation in the new notation becomes

$$\tfrac{-\hbar^2}{2m}\hat{\partial}_h\hat{\partial}^h\hat{\psi}(\hat{\mathrm{x}})+\hat{V}(\hat{\mathrm{x}})\hat{\psi}(\hat{\mathrm{x}}) = E\hat{\psi}(\hat{\mathrm{x}}).$$

2 Preliminaries

2.1 Twisting $H = U\mathbf{g}$ to a noncocommutative Hopf algebra $\hat{H}$

The Universal Enveloping $*$-Algebra (UEA) $H := U\mathbf{g}$ of the Lie algebra $\mathbf{g}$ of any Lie group G is a Hopf $*$-algebra. First, we briefly recall what this means. Let

$$\begin{aligned}
&\varepsilon(\mathbf{1}) = 1, \qquad && \Delta(\mathbf{1}) = \mathbf{1}\otimes\mathbf{1}, \qquad && S(\mathbf{1}) = \mathbf{1}, && \\
&\varepsilon(g) = 0, \qquad && \Delta(g) = g\otimes\mathbf{1} + \mathbf{1}\otimes g, \qquad && S(g) = -g, \qquad && \text{if } g \in \mathbf{g};
\end{aligned}$$

ε, Δ are extended to all of H as $*$-algebra maps, S as a $*$-antialgebra map:

$$\begin{aligned}
&\varepsilon : H \to \mathbb{C}, \qquad && \varepsilon(ab) = \varepsilon(a)\varepsilon(b), \qquad && \varepsilon(a^*) = [\varepsilon(a)]^*, \\
&\Delta : H \to H\otimes H, \qquad && \Delta(ab) = \Delta(a)\Delta(b), \qquad && \Delta(a^*) = [\Delta(a)]^{*\otimes *}, \\
&S : H \to H, \qquad && S(ab) = S(b)S(a), \qquad && S\left\{[S(a^*)]^*\right\} = a.
\end{aligned} \tag{5}$$

The extensions of ε, Δ, S are unambiguous, as $\varepsilon(g) = 0$, $\Delta\big([g,g']\big) = \big[\Delta(g), \Delta(g')\big]$, $S\big([g,g']\big) = \big[S(g'), S(g)\big]$ if $g, g' \in \mathbf{g}$. The maps ε, Δ, S are the abstract operations by which one constructs the trivial representation, the tensor product of any two representations and the contragredient of any representation, respectively. $H = U\mathbf{g}$ equipped with $*, \varepsilon, \Delta, S$ is a Hopf $*$-algebra.

Second, we deform this Hopf algebra. Let $\hat{H} = H[[\lambda]]$. Given a *twist* [11] (see also [21, 9]), i.e. an element $\mathcal{F} \in (H\otimes H)[[\lambda]]$ fulfilling

$$\mathcal{F} = \mathbf{1}\otimes\mathbf{1} + O(\lambda), \qquad (\epsilon\otimes\mathrm{id})\mathcal{F} = (\mathrm{id}\otimes\epsilon)\mathcal{F} = \mathbf{1}, \tag{6}$$

$$(\mathcal{F}\otimes\mathbf{1})[(\Delta\otimes\mathrm{id})(\mathcal{F})] = (\mathbf{1}\otimes\mathcal{F})[(\mathrm{id}\otimes\Delta)(\mathcal{F})] =: \mathcal{F}^3, \tag{7}$$

we shall call $H_s \subseteq H$ the smallest Hopf $*$-subalgebra such that $\mathcal{F} \in (H_s\otimes H_s)[[\lambda]]$ and

$$\sum_I \mathcal{F}_I^{(1)}\otimes\mathcal{F}_I^{(2)} := \mathcal{F}, \qquad \sum_I \overline{\mathcal{F}}_I^{(1)}\otimes\overline{\mathcal{F}}_I^{(2)} := \mathcal{F}^{-1}, \qquad \beta := \sum_I \mathcal{F}_I^{(1)} S\left(\mathcal{F}_I^{(2)}\right) \in H_s. \tag{8}$$

Without loss of generality λ can be assumed real; for our purposes $\mathcal{F}$ is *unitary* ($\mathcal{F}^{*\otimes *} = \mathcal{F}^{-1}$), implying that also β is ($\beta^* = \beta^{-1}$). Extending the product, $*, \Delta, \varepsilon, S$ linearly to the formal power series in λ and setting

$$\hat{\Delta}(g) := \mathcal{F}\Delta(g)\mathcal{F}^{-1}, \qquad \hat{S}(g) := \beta\, S(g)\beta^{-1}, \qquad \mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1}, \tag{9}$$

one finds that the analogs of conditions (5) are satisfied and therefore $(\hat{H}, *, \hat{\Delta}, \varepsilon, \hat{S})$ is a Hopf $*$-algebra deformation of the initial one and has a unitary triangular structure $\mathcal{R}$ (i.e. $\mathcal{R}^{-1} = \mathcal{R}_{21} = \mathcal{R}^{*\otimes *}$). While H is cocommutative, i.e. $\tau\circ\Delta(g) = \Delta(g)$ where τ is the flip operator $[\tau(a\otimes b) = b\otimes a]$, $\hat{H}$ is triangular noncocommutative i.e. $\tau\circ\hat{\Delta}(g) = \mathcal{R}\Delta(g)\mathcal{R}^{-1}$. Correspondingly, $\hat{\Delta}, \hat{S}$ replace Δ, S in the construction of the tensor product of any two representations and the contragredient of any representation, respectively. Drinfel'd has shown [11] that any triangular deformation of the Hopf algebra H can be obtained in this way (up to isomorphisms).

Eq. (7), (9) imply the generalized intertwining relation $\hat{\Delta}^{(n)}(g)=\mathcal{F}^n\Delta^{(n)}(g)(\mathcal{F}^n)^{-1}$ for the iterated coproduct. By definition

$$\hat{\Delta}^{(n)}:\hat{H}\to\hat{H}^{\otimes n},\qquad \Delta^{(n)}:H[[\lambda]]\to(H)^{\otimes n}[[\lambda]],\qquad \mathcal{F}^n\in(H_s)^{\otimes n}[[\lambda]]$$

reduce to $\hat{\Delta},\Delta,\mathcal{F}$ for $n=2$, whereas for $n>2$ they can be defined recursively as

$$\begin{aligned}&\hat{\Delta}^{(n+1)}=(\mathrm{id}^{\otimes^{n-1}}\otimes\hat{\Delta})\circ\hat{\Delta}^{(n)},\qquad \Delta^{(n+1)}=(\mathrm{id}^{\otimes(n-1)}\otimes\Delta)\circ\Delta^{(n)},\\&\mathcal{F}^{n+1}=(\mathbf{1}^{\otimes(n-1)}\otimes\mathcal{F})[(\mathrm{id}^{\otimes(n-1)}\otimes\Delta)\mathcal{F}^n].\end{aligned}\tag{10}$$

The result for $\hat{\Delta}^{(n)},\mathcal{F}^n$ is the same if in definitions (10) we iterate the coproduct on a different sequence of tensor factors [coassociativity of $\hat{\Delta}$; this follows from the coassociativity of Δ and the cocycle condition (7)]; for instance, for $n\!=\!3$ this amounts to (7) and $\hat{\Delta}^{(3)}\!=\!(\hat{\Delta}\otimes\mathrm{id}\,)\circ\hat{\Delta}$. For any $g\in H[[h]]=\hat{H}$ we shall use the Sweedler notations

$$\Delta^{(n)}(g)=\sum_I g^I_{(1)}\otimes g^I_{(2)}\otimes...\otimes g^I_{(n)},\qquad\qquad \hat{\Delta}^{(n)}(g)=\sum_I g^I_{(\hat{1})}\otimes g^I_{(\hat{2})}\otimes...\otimes g^I_{(\hat{n})}.$$

For the Euclidean or Poincaré group, calling $P_\mu,M_{\mu\nu}$ respectively the generators of translations, homogenous transformations and adopting the Moyal twist $(2)_2$ one finds $\beta=\mathbf{1}$, $\hat{S}=S$ and

$$\begin{aligned}&\hat{\Delta}(P_\mu)=P_\mu\otimes\mathbf{1}+\mathbf{1}\otimes P_\mu=\Delta(P_\mu),\\&\hat{\Delta}(M_\omega)=M_\omega\otimes\mathbf{1}+\mathbf{1}\otimes M_\omega+P_\mu([\omega,\theta])^{\mu\nu}\otimes P_\nu\neq\Delta(M_\omega).\end{aligned}$$

where $M_\omega\!=\!\omega^{\mu\nu}M_{\mu\nu}$. The Hopf subalgebra of translations is undeformed!

2.2 Twisting H-module $*$-algebras

We recall that a $*$-algebra $\mathcal{A}$ over $\mathbb{C}$ is defined to be a left H-module $*$-algebra if there exists a $\mathbb{C}$-bilinear map $(g,a)\!\in\! H\times\mathcal{A}\to g\triangleright a\!\in\!\mathcal{A}$, called (left) action, such that

$$(gg')\triangleright a=g\triangleright(g'\triangleright a),\qquad (g\triangleright a)^*=[S(g)]^*\triangleright a^*,\qquad g\triangleright(ab)=\sum_I\big(g^I_{(1)}\triangleright a\big)\big(g^I_{(2)}\triangleright b\big)\,.\tag{11}$$

Given such an $\mathcal{A}$, let $V\big(\mathcal{A}\big)$ the vector space underlying $\mathcal{A}$. $V\big(\mathcal{A}\big)[[\lambda]]$ gets a $\hat{H}$-module $*$-algebra $\mathcal{A}_\star$ when endowed with the product and $*$-structure

$$a\star a':=\sum_I\Big(\overline{\mathcal{F}}^{(1)}_I\triangleright a\Big)\Big(\overline{\mathcal{F}}^{(2)}_I\triangleright a'\Big)\,,\qquad\qquad a^{*_\star}:=S(\beta)\triangleright a^*.\tag{12}$$

In fact, $\star$ is associative by (7), fulfills $(a\star a')^{*_\star}=a'^{*_\star}\star a^{*_\star}$ and

$$g\triangleright(a\star a')=\textstyle\sum_I\Big[g^I_{(\hat{1})}\triangleright a\Big]\star\Big[g^I_{(\hat{2})}\triangleright a'\Big]\,.\tag{13}$$

This is mostly used to deform abelian $\mathcal{A}$, but works even if $\mathcal{A}$ is non-abelian.
Note that the $\star$ *is ineffective if a or a' is H^s-invariant:*

$$g\triangleright a=\epsilon(g)a\quad\text{or}\quad g\triangleright a'=\epsilon(g)a'\quad\forall g\in H^s\qquad\Rightarrow\qquad a\star a'=aa'.\tag{14}$$

Given H-module $*$-algebras $\mathcal{A},\mathcal{B}$, also $\mathcal{A}\otimes\mathcal{B}$ is, so (12) makes $V(\mathcal{A}\otimes\mathcal{B})$ into a $\hat{H}$-module $*$-algebra $(\mathcal{A}\otimes\mathcal{B})_\star$. Defining a bilinear map $\otimes_\star$ by $a\otimes_\star b:=(a\otimes\mathbf{1}_\mathcal{B})\star(\mathbf{1}_\mathcal{A}\otimes b)$ one finds

$$(a\otimes_\star b)\star(a'\otimes_\star b') = \textstyle\sum_I a\star(\mathcal{R}_I^{(2)}\triangleright a')\otimes_\star(\mathcal{R}_I^{(1)}\triangleright b)\star b', \tag{15}$$

so $\otimes_\star$ is the *(involutive) braided tensor product* associated to $\mathcal{R}$, and $(\mathcal{A}\otimes\mathcal{B})_\star=\mathcal{A}_\star\otimes_\star\mathcal{B}_\star$.

If $\mathcal{A}$ is defined by generators a_i and relations, then also $\mathcal{A}_\star$ is, with the same Poincaré-Birkhoff-Witt series [17]. One can define a *linear map* $\wedge: f\in\mathcal{A}\to\hat{f}\in\mathcal{A}_\star$ by the equation

$$f(a_1,a_2,...)\star=\hat{f}(a_1\star,a_2\star,...) \qquad \text{in } V(\mathcal{A})=V(\mathcal{A}_\star). \tag{16}$$

This is the generalization of (4). We shall often change notation and replace: $a_i\star a_j\rightsquigarrow\hat{a}_i\hat{a}_j$, $\hat{f}(a_i\star)\rightsquigarrow\hat{f}(\hat{a}_i)$, $\mathcal{A}_\star\rightsquigarrow\widehat{\mathcal{A}}$, $*_\star\rightsquigarrow\hat{*}$ etc.

2.3 Application to differential and integral calculi

If $\mathcal{X}$ is the algebra of smooth functions on a manifold X and Ξ the Lie algebra of smooth vector fields on X, one finds that $\mathcal{X}$ and $\mathcal{D}=U\Xi\ltimes\mathcal{X}$ (the algebra of smooth differential operators on X) are $U\Xi$-module $*$-algebras. Each twist $\mathcal{F}\in(U\Xi\otimes U\Xi)[[\lambda]]$ generates a$\star$-deformation [3]

$$\mathcal{X}\overset{(12)}{\longrightarrow}\mathcal{X}_\star\sim\widehat{\mathcal{X}}, \qquad\qquad \mathcal{D}\overset{(12)}{\longrightarrow}\mathcal{D}_\star\sim\widehat{\mathcal{D}}.$$

Assuming X is Riemannian, let G be its group of isometries, $H=U\mathbf{g}$, $d\nu$ the G-invariant volume form on X. Fixed a $\mathcal{F}\in(H\otimes H)[[\lambda]]$, the invariance of integration [i.e. $\int_X d\nu(g\triangleright f)=\epsilon(g)\int_X d\nu f$] implies for the corresponding $\star$-product

$$\int_X d\nu(\mathrm{x})f(\mathrm{x})\star h(\mathrm{x})=\int_X d\nu(\mathrm{x})f(\mathrm{x})[\beta^{-1}\triangleright h(\mathrm{x})]=\int_X d\nu(\mathrm{x})[S(\beta^{-1})\triangleright f(\mathrm{x})]h(\mathrm{x}). \tag{17}$$

The invariance of the Laplacian ∇^2 implies $\nabla^2\star=\nabla^2$ by (14); moreover, ∇^2 itself can be expressed as a $\star$-product of two vector fields, e.g. on $X=\mathbb{R}^m$ $\nabla^2=\partial_h\star\partial'_h$, where $\partial'_h=S(\beta)\triangleright\partial_h$. As a result, the kinetic part of the Hamitlonian (3) remains undeformed. For the Moyal $\star$-product on $X=\mathbb{R}^m$ it is $\beta=\mathbf{1}$, whence $\int_X d\nu f\star h=\int_X d\nu f h$ and $\partial'_h=\partial_h$.

We now further assume that X admits global coordinates x^h (so that $\mathcal{X}$ is generated by the x^h) and that the map $\wedge: f\in\mathcal{X}[[\lambda]]\to\hat{f}\in\mathcal{X}_\star$ is well-defined (so that $\mathcal{X}_\star$ is generated by the $\mathrm{x}^h\star$). This is the case e.g. if $X\subseteq\mathbb{R}^m$ is an algebraic manifold symmetric under a subgroup $G\subseteq IGL(m)$, and $\mathcal{F}\in(H\otimes H)[[\lambda]]$, with $H=U\mathbf{g}$ and $\mathbf{g}=Lie(G)$. Then one can define also a $\hat{H}$-invariant "integration over $\hat{X}$" $\int_{\hat{X}}d\hat{\nu}(\hat{\mathrm{x}})$ such that for each $f\in\mathcal{X}$

$$\int_{\hat{X}}d\hat{\nu}(\hat{\mathrm{x}})\hat{f}(\hat{\mathrm{x}})=\int_X d\nu(\mathrm{x})f(\mathrm{x}). \tag{18}$$

We shall call $\wedge^n$ the analogous maps $\wedge^n: f\in\mathcal{X}^{\otimes n}[[\lambda]]\to\hat{f}\in(\mathcal{X}^{\otimes n})_\star$. The previous two equations generalize to integration over n independent x-variables.

2.4 Application to the Heisenberg/Clifford algebra $\mathcal{A}^{\pm}$

The covariance under a Lie group G of a quantum theory describing a species of bosons (resp. fermions) implies that the associated Heisenberg algebra $\mathcal{A}^{+}$ (resp. Clifford algebra $\mathcal{A}^{-}$) is a $U\mathbf{g}$-module $*$-algebra (e.g., for non-relativistic quantum mechanics on $\mathbb{R}^3$ G is the Euclidean group or its extension the Galilei group, for the relativistic theory on Minkowski space G is the Poincaré group). In this subsection we first recall how this happens, then $\star$-deform $\mathcal{A}^{\pm}$; we describe the quantum system abstractly (i.e. in terms of bra, kets, abstract operators). As the Lie group (of *active* transformations) G is unitarily implemented on the Hilbert space of the system, the action of $H = U\mathbf{g}$ will be defined on a dense subspace, in particular on a pre-Hilbert space $\mathcal{H}$ of the one-particle sector, on which it will be denoted as ρ: $g\triangleright =: \rho(g)\in\mathcal{O}:=\mathrm{End}(\mathcal{H}) \quad \forall g\in H$.
The pre-Hilbert space of n bosons (resp. fermions) is described by the completely symmetrized (resp. antisymmetrized) tensor product $\mathcal{H}_{+}^{\otimes n}$ (resp. $\mathcal{H}_{-}^{\otimes n}$), which is a H-$*$-submodule of $\mathcal{H}^{\otimes n}$. Denoting as $|0\rangle$ the vacuum state, the bosonic (resp. fermionic) Fock space is defined as the closure $\overline{\mathcal{H}_{\pm}^{\infty}}$ of

$$\mathcal{H}_{\pm}^{\infty} := \left\{\text{finite sequences } (s_0, s_1, s_2, ...) \in \mathbb{C}|0\rangle \oplus \mathcal{H} \oplus \mathcal{H}_{\pm}^{2} \oplus ...\right\}$$

(finite means that there exists an integer $l \geq 0$ such that $s_n = 0$ for all $n \geq l$). As usual for any orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$ of $\mathcal{H}$ we can define an associated set of creation, annihilation operators for bosons, fermions fulfilling the Canonical (anti)Commutation Relations (CCR)

$$[a^i, a^j]_{\mp} = 0, \qquad [a^+_i, a^+_j]_{\mp} = 0, \qquad [a^i, a^+_j]_{\mp} = \delta^i_j \mathbf{1}. \tag{19}$$

The H-invariance of the vacuum $|0\rangle$ implies that creation and annihilation operators a^+_i, a^i must transform as the vectors $e_i = a^+_i|0\rangle$ and $\langle e_i, \cdot\rangle = \langle 0|a^i$ respectively:

$$g \triangleright e_i = \rho^j_i(g) e_j \qquad \Rightarrow \qquad g \triangleright a^+_i = \rho^j_i(g) a^+_j, \qquad g \triangleright a^i = \rho^{\vee}{}^j_i(g) a^j = \rho^i_j\big[S(g)\big] a^j \tag{20}$$

($\rho^{\vee} = \rho^T \circ S$ is the contragredient of ρ). Therefore $\mathcal{A}^{\pm}$ is a H-module $*$-algebra because the $\mathbf{g}$-action (extended to products as a derivation) is compatible with the (19).

Applying the deformation procedure one obtains $\hat{H}$-module $*_{\star}$-algebras $\mathcal{A}^{\pm}_{\star}$. The generators a^+_i, $a'^i := a^{+*_{\star}}_i = \rho^i_j(\beta) a^j$ fulfill the $\star$-commutation relations

$$\begin{aligned}
a'^i\star a'^j &= \pm R^{ij}_{vu} a'^u\star a'^v, & \qquad & \hat{a}'^i\hat{a}'^j = \pm R^{ij}_{vu}\hat{a}'^u\hat{a}'^v,\\
a^+_i\star a^+_j &= \pm R^{vu}_{ij} a^+_u\star a^+_v, & \Leftrightarrow \qquad & \hat{a}^+_i\hat{a}^+_j = \pm R^{vu}_{ij}\hat{a}^+_u\hat{a}^+_v,\\
a'^i\star a^+_j &= \delta^i_j \mathbf{1}_{\mathcal{A}} \pm R^{ui}_{jv} a^+_u\star a'^v, & \qquad & \hat{a}'^i\hat{a}^+_j = \delta^i_j \mathbf{1}_{\hat{\mathcal{A}}} \pm R^{ui}_{jv}\hat{a}^+_u\hat{a}'^v,
\end{aligned} \tag{21}$$

where $R := (\rho\otimes\rho)(\mathcal{R})$. The a'^i transform according to the rule of the *twisted* contragredient representaton: $g \triangleright a'^i = \rho^i_j\big[\hat{S}(g)\big] a'^j$. Equivalently, $\widehat{\mathcal{A}}^{\pm} \sim \mathcal{A}^{\pm}_{\star}$ has generators $\hat{a}^+_i, \hat{a}^i$ fulfilling $\hat{a}^{+\hat{*}}_i = \hat{a}'^i$ and the rhs(21). Such a general class of equivariantly deformed Heisenberg/Clifford algebras was introduced in Ref. [13]. Up to normalization of R the relations at rhs(21) are actually identical to the ones defining the older q-deformed Heisenberg algebras of [20, 19, 23], based on a quasitriangular $\mathcal{R}$ in (only) the *fundamental* representation of $H = U_q su(N)$ (i.e. $i, j, u, v\in\{1, ..., N\}$).

What are the $*$-representations of $\widehat{\mathcal{A}}^{\pm}$? Is there a Fock type one? Yes, on the *undeformed* Fock space of bosons/fermions. The important consequence is that **(21) are compatible with Bose/Fermi statistics** [14]. The simplest explanation of this is that one can "realize" $\hat{a}_i^+, \hat{a}'^i$ as "dressed" elements $\check{a}_i^+, \check{a}'^i$ in $\mathcal{A}^{\pm}[[\lambda]]$ fulfilling (21) and hermitean conjugate to each other [13]:

$$\check{a}_i^+ = \sum_I \left(\overline{\mathcal{F}}_I^{(1)} \triangleright a_i^+\right) \sigma\left(\overline{\mathcal{F}}_I^{(2)}\right), \qquad \check{a}'^i = \sum_I \left(\overline{\mathcal{F}}_I^{(1)} \triangleright a'^i\right) \sigma\left(\overline{\mathcal{F}}_I^{(2)}\right). \tag{22}$$

In (22) we have used the $*$-algebra map $\sigma : H[[\lambda]] \to \mathcal{A}^{\pm}[[\lambda]]$, which is defined by setting on the generators $\sigma(\mathbf{1}_H) = \mathbf{1}_{\mathcal{A}}$, $\sigma(g) = (g \triangleright a_j^+) a^j$ if $g \in \mathbf{g}$; another characterizing property is that

$$g \triangleright a = \sum_I \sigma\left(g^I_{(1)}\right) a\, \sigma\left(g^I_{(2)}\right) \qquad \forall g \in H, \quad a \in \mathcal{A}^{\pm}.$$

For $\mathbf{g} = su(2)$ σ is the well-known Jordan-Schwinger realization of $Usu(2)$. For Moyal deformation of $\mathbb{R}^m$, with generalized basis $\{e_{\mathrm{p}}\}$, $P^h e_{\mathrm{p}} = p^h e_{\mathrm{p}}$, (22) reduces to

$$\check{a}_{\mathrm{p}}^+ = a_{\mathrm{p}}^+ e^{-\frac{i}{2} p\theta\sigma(P)}, \qquad \check{a}^{\mathrm{p}} = a^{\mathrm{p}} e^{\frac{i}{2} p\theta\sigma(P)}, \qquad \sigma(P^h) := \int d^m p\, p^h a_{\mathrm{p}}^+ a^{\mathrm{p}}.$$

The latter formulae have already appeared in the literature (see [17] for a list of references). Provided the λ-power series entailed in (22) converge, $\check{a}_i^+, \check{a}'^i$ are well-defined operators on the Fock space, providing on the latter also a representation of $\widehat{\mathcal{A}}^{\pm}$. One can also show [17] that this is the only representation of $\widehat{\mathcal{A}}^{\pm}$ of Fock type.

3 Non-relativistic second quantization

3.1 Twisting quantum mechanics in configuration space

Dealing with a wave-mechanical description of a system of quantum particles means that the state vectors s's are described by wavefunctions ψ's on X and the abstract operators by differential or more generally integral operators on the ψ's. For simplicity we choose $X = \mathbb{R}^3$, consider spinless particles and derive consequences from the covariance of the description first under the Euclidean group G (thought as a group of *active* space-symmetry transformations), then under the whole Galilei group G'. Going to the infinitesimal form, all elements $H = U\mathbf{g}$ will be well-defined differential operators e.g. on the pre-Hilbert space $\mathcal{S}(\mathbb{R}^3)$, so we can choose $\mathcal{X}$ as a dense subspace $\mathcal{X} \subseteq \mathcal{S}(\mathbb{R}^3)$ (to be specified later) and tailor the 1-particle pre-Hilbert space $\mathcal{H}$ and the algebra of endomorphisms $\mathcal{O} := \mathrm{End}(\mathcal{H})$ as respectively isomorphic to $\mathcal{X}, \mathcal{E} := \mathrm{End}(\mathcal{X})$, by definition; we shall call the isomorhisms $\kappa, \tilde{\kappa}$. [One reason why we do not identify $\mathcal{H}$ with $\mathcal{X}$ is that we wish to introduce a *realization* (i.e. representation) of the *same* state $s \in \mathcal{H}$ of the quantum system also by a noncommutative wavefunction.] Summarizing, there exists a (frame-dependent) $H = U\mathbf{g}$ *-equivariant configuration space realization of* $\{\mathcal{H}, \mathcal{O}\}$ *on* $\{\mathcal{X}, \mathcal{E}\}$, i.e.

1. there exists a H-equivariant, unitary transformation $\kappa : s \in \mathcal{H} \leftrightarrow \psi_s \in \mathcal{X}$,

$$g \triangleright \psi_s = \psi_{g \triangleright s}, \qquad \langle s|v\rangle = \int_X d\nu\, [\psi_s(\mathrm{x})]^* \psi_v(\mathrm{x}). \tag{23}$$

2. $\kappa(Os)=\tilde{\kappa}(O)\kappa(s)$ for any $s\in\mathcal{H}$ defines a H-equivariant map $\tilde{\kappa}:O\in\mathcal{O}\leftrightarrow D_O\in\mathcal{E}$, and $\mathcal{D}\subset\mathcal{E}$.

This implies for a system of n distinct particles (resp. n bosons/fermions) on X:

1. $\kappa^{\otimes n}:\mathcal{H}^{\otimes n}\leftrightarrow\mathcal{X}^{\otimes n}$ (resp. the restrictions $\kappa^{\otimes n}:\mathcal{H}^{\otimes n}_{\pm}\leftrightarrow\mathcal{X}^{\otimes n}_{\pm}$) are H-equivariant unitary transformations.

2. $\tilde{\kappa}^{\otimes n}:\mathcal{O}^{\otimes n}\leftrightarrow\mathcal{E}^{\otimes n}$ (resp. the restriction $\tilde{\kappa}^{\otimes n}:\mathcal{O}^{\otimes n}_{+}\leftrightarrow\mathcal{E}^{\otimes n}_{+}$) are H-equivariant maps.

For any twist $\mathcal{F}\in(H\otimes H)[[\lambda]]$ and the associated $\star$-deformation one finds

$$\begin{aligned}\langle s,v\rangle &= \int_X d\nu(\mathrm{x}_1)...\int_X d\nu(\mathrm{x}_n)[\psi_s(\mathrm{x}_1,...,\mathrm{x}_n)]^*\psi_v(\mathrm{x}_1,...,\mathrm{x}_n)\\ &\overset{(17),(12)}{=} \int_X d\nu(\mathrm{x}_1)...\int_X d\nu(\mathrm{x}_n)[\psi_s(\mathrm{x}_1,...,\mathrm{x}_n)]^{*_\star}\star\psi_v(\mathrm{x}_1,...,\mathrm{x}_n).\end{aligned} \tag{24}$$

If in addition $\mathcal{F}$ is such that one can define the map $\wedge$ (see section **??**.3), we introduce noncommutative wavefunctions $\hat{\psi}=\wedge^n(\psi)$. Then the previous equation becomes

$$\langle s,v\rangle=\int_{\hat{X}}d\hat{\nu}(\hat{\mathrm{x}}_1)...\int_{\hat{X}}d\hat{\nu}(\hat{\mathrm{x}}_n)[\hat{\psi}_s(\hat{\mathrm{x}}_1,...,\hat{\mathrm{x}}_n)]^{\hat{*}}\hat{\psi}_v(\hat{\mathrm{x}}_1,...,\hat{\mathrm{x}}_n). \tag{25}$$

The map $\wedge^n:\psi_s\in\mathcal{X}^{\otimes n}\to\hat{\psi}_s\in(\mathcal{X}^{\otimes n})_\star$ is thus unitary and $\hat{H}$-equivariant.
The action of the symmetric group S_n on $(\mathcal{X}^{\otimes n})_\star$ is obtained by "pull-back" from that on $\mathcal{X}^{\otimes n}$: a permutation $\tau\in S_n$ represented on $\mathcal{X}^{\otimes n}$,$(\mathcal{X}^{\otimes n})_\star$ resp. by the permutation operator $\mathcal{P}_\tau$ and the "twisted permutation operator" $\mathcal{P}^F_\tau=\wedge^n\mathcal{P}_\tau[\wedge^n]^{-1}$. Thus, $(\mathcal{X}^{\otimes n}_{\pm})_\star,(\mathcal{E}^{\otimes n}_{+})_\star$ are (anti)symmetric up to the similarity transformation $\wedge^n$ (cf. [14]).

Let $\hat{\kappa}^n:=\wedge^n\kappa^{\otimes n}$, $\widehat{\tilde{\kappa}}^n(\cdot):=\wedge^n[\tilde{\kappa}^{\otimes n}(\cdot)][\wedge^n]^{-1}$. The restrictions $\hat{\kappa}^n\lceil_{\mathcal{H}^{\otimes n}_{\pm}}$, $\widehat{\tilde{\kappa}}^n\lceil_{\mathcal{O}^{\otimes n}_{+}}$ define a (frame-dependent) $\hat{H}$*-equivariant, noncommutative configuration space realization* of $\{\mathcal{H}^{\otimes n}_{\pm},\mathcal{O}^{\otimes n}_{+}\}$ on $\{(\mathcal{X}^{\otimes n}_{\pm})_\star,(\mathcal{E}^{\otimes n}_{+})_\star\}$ (or, changing notation, on $\{\widehat{\mathcal{X}^{\otimes n}_{\pm}},\widehat{\mathcal{E}^{\otimes n}_{+}}\}$).

3.2 Quantum fields in the Schrödinger picture

Given a basis $\{e_i\}_{i\in\mathbb{N}}\subset\mathcal{H}$, let $\varphi_i=\kappa(e_i)$. The **nonrelativistic field operator** and its hermitean conjugate

$$\varphi(\mathrm{x}):=\varphi_i(\mathrm{x})a^i,\qquad\qquad \varphi^*(\mathrm{x})=\varphi_i^*(\mathrm{x})a_i^+ \tag{26}$$

(infinite sum over i) are operator-valued distributions fulfilling the commutation relations

$$[\varphi(\mathrm{x}),\varphi(\mathrm{y})]_\mp=\mathrm{h.c.}=0,\qquad [\varphi(\mathrm{x}),\varphi^*(\mathrm{y})]_\mp=\varphi_i(\mathrm{x})\varphi_i^*(\mathrm{y})=\delta(\mathrm{x}-\mathrm{y}) \tag{27}$$

($\mp$ for bosons/fermions). The *field $*$-algebra* Φ can be defined as the span of the normal ordered monomials

$$\varphi^*(\mathrm{x}_1)....\varphi^*(\mathrm{x}_m)\varphi(\mathrm{x}_{m+1})...\varphi(\mathrm{x}_n) \tag{28}$$

($\mathrm{x}_1,...,\mathrm{x}_n$ are independent points). So $\Phi\subset\Phi^e:=\mathcal{A}^\pm\otimes(\bigotimes_{i=1}^\infty\mathcal{X}')$ (here the 1st, 2nd,... tensor factor $\mathcal{X}'$ is the space of distributions depending on $\mathrm{x}_1,\mathrm{x}_2$,...); the dependence of (28) on x_h is trivial for $h>n$. Φ^e is a huge H-module $*$-algebra: a_i^+,φ_i transform as e_i,

and a^i, φ_i^* transform as $\langle e_i, \cdot\rangle$. The CCR (19) of $\mathcal{A}^\pm$ are the only nontrivial commutation relations in Φ^e.

The key property is that φ, φ^* are basis-independent, i.e. **invariant under the group $U(\infty)$ of unitary transformations of $\{e_i\}_{i\in\mathbb{N}}$**, in particular under the subgroup G of Euclidean transformations (transformations of the states e_i obatined by translations or rotations of the 1-particle system), or (in infinitesimal form) **under $U\mathbf{g}$**: $g \triangleright \varphi(\mathrm{x}) = \epsilon(g)\varphi(\mathrm{x})$.

As a consequence, if we apply the $\star$-deformation with the above $\mathcal{F}$ we deform $U\mathbf{g} \to \widehat{U\mathbf{g}}$, $Uu(\infty) \to \widehat{Uu(\infty)}$ and the associated module $*$-algebra $\Phi^e \overset{(12)}{\longrightarrow} \Phi^e_\star \sim \widehat{\Phi^e}$, but find

$$\varphi(\mathrm{x})\star\omega = \varphi(\mathrm{x})\omega, \qquad \omega\star\varphi(\mathrm{x}) = \omega\,\varphi(\mathrm{x}), \qquad \& \quad \text{h. c.}, \qquad \forall\omega\in V(\Phi^e)[[\lambda]], \tag{29}$$

because of (14). Since $\epsilon(\beta) = 1$ and the definition $a'^i := a_i^{+*\star} = S(\beta) \triangleright a^i$ imply

$$\varphi(\mathrm{x}) = \varphi_i(\mathrm{x}) \star a'^i, \qquad \varphi^*(\mathrm{x}) = \varphi^{*\star}(\mathrm{x}) = a_i^+ \star \varphi_i^{*\star}(\mathrm{x}), \tag{30}$$

and $\varphi_i(\mathrm{x})\varphi_i^*(\mathrm{y}) = \varphi_i(\mathrm{x}) \star \varphi_i^{*\star}(\mathrm{y})$, in $\Phi^e_\star$ the CCR (27) become

$$[\varphi(\mathrm{x}) \overset{\star}{,} \varphi(\mathrm{y})]_\mp = h.c. = 0, \qquad [\varphi(\mathrm{x}) \overset{\star}{,} \varphi^{*\star}(\mathrm{y})]_\mp = \varphi_i(\mathrm{x}) \star \varphi_i^{*\star}(\mathrm{y}) \tag{31}$$

(here $[A \overset{\star}{,} B]_\mp := A \star B \mp B \star A$). $\Phi^e_\star$ is a huge $\widehat{U\mathbf{g}}$-module [and also $\widehat{Uu(\infty)}$-module] $*$-algebra.

The unitary map $\kappa^n_\pm : s\in\mathcal{H}_\pm^{\otimes n} \leftrightarrow \psi_s\in\mathcal{X}_\pm^{\otimes n}$ and its inverse can be computed using the field

$$\begin{aligned}
\psi_s(\mathrm{x}_1,\ldots,\mathrm{x}_n) &= \tfrac{1}{\sqrt{n!}} \left\langle [\varphi^*(\mathrm{x}_1)\ldots\varphi^*(\mathrm{x}_n)|0\rangle]\,, s\right\rangle, \\
s &= \tfrac{1}{\sqrt{n!}} \int_X d\nu(\mathrm{x}_1)\ldots\int_X d\nu(\mathrm{x}_n)\varphi^*(\mathrm{x}_1)\ldots\varphi^*(\mathrm{x}_n)|0\rangle\; \psi_s(\mathrm{x}_1,\ldots,\mathrm{x}_n).
\end{aligned} \tag{32}$$

For $s \in \mathcal{H}^{\otimes n}$, $\psi_s \in \mathcal{X}^{\otimes n}$ the rhs of these equations give projections $\pi^n_\pm : \mathcal{H}^{\otimes n} \to \mathcal{X}_\pm^{\otimes n}$ and $\Pi^n_\pm : \mathcal{X}^{\otimes n} \to \mathcal{H}_\pm^{\otimes n}$. Analogous properties hold also for the deformed counterparts (see [17]).

3.3 Equations of motion and Heisenberg picture

Assume the n-particle wavefunction $\psi^{(n)}$ fulfills the Schrödinger equation (3) if $n=1$, and

$$i\hbar\tfrac{\partial}{\partial t}\psi^{(n)} = \mathsf{H}_\star^{(n)}\psi^{(n)}, \qquad \mathsf{H}_\star^{(n)} := \sum_{h=1}^{n} \mathsf{H}_\star^{(1)}(\mathrm{x}_h, \partial_h, t) + \sum_{h<k} W(\rho_{hk})\star \tag{33}$$

if $n \geq 2$; here the time coordinate t remains "commuting", and the 2-body interaction W depends only on the (invariant) distance ρ_{hk} between $\mathrm{x}_h, \mathrm{x}_k$. $\mathsf{H}_\star^{(n)}$ will be hermitean provided $\mathsf{H}^{(1)}$ is and $\beta\triangleright\mathsf{H}^{(1)} = \mathsf{H}^{(1)}$, as we shall assume. In general this is a $\star$-differential, pseudodifferential equation, preserving the (anti)symmetry of $\psi^{(n)}$. The Fock space Hamiltonian

$$\mathsf{H}_\star(\varphi) = \int_X d\nu(\mathrm{x})\varphi^{*\star}(\mathrm{x}) \star \mathsf{H}_\star^{(1)}\varphi(\mathrm{x})\star + \int_X d\nu(\mathrm{x})\int_X d\nu(\mathrm{y})\varphi^{*\star}(\mathrm{y})\star\varphi^{*\star}(\mathrm{x})\star W(\rho_{\mathrm{xy}})\star\varphi(\mathrm{x})\star\varphi(\mathrm{y})\star$$

annihilates the vacuum, commutes with the number-of-particles operator $\mathbf{n} := a_i^+ \star a^i$ and its restriction to $\mathcal{H}_\pm^{\otimes n}$ coincides with $\mathsf{H}_\star^{(n)}$ up to the unitary transformation $\tilde{\kappa}^{\otimes n}$. As in the

undeformed setting, formulating the dynamics on the Fock space allows to consider also more general Hamiltonians $\mathsf{H}_\star$, which do not commute with $\mathbf{n}$.

The Heisenberg field $\varphi^H_\star(\mathrm{x},t) := [U(t)]^{*\star}\varphi(\mathrm{x})U(t)$ fulfills the equal time commutation relations

$$
\begin{array}{ll}
[\varphi^H_\star(\mathrm{x},t) \overset{\star}{,} \varphi^H_\star(\mathrm{y},t)]_\mp = h.c. = 0, & [\varphi^H_\star(\mathrm{x},t) \overset{\star}{,} \varphi^{H*_\star}_\star(\mathrm{y},t)]_\mp = \varphi_i(\mathrm{x})\star\varphi_i^{*_\star}(\mathrm{y}) \\
i\hbar\frac{\partial}{\partial t}\varphi^H_\star = [\varphi^H_\star, \mathsf{H}_\star] \qquad \Rightarrow & i\hbar\frac{\partial\varphi^H_\star}{\partial t} = \mathsf{H}^{(1)}_\star\varphi^H_\star \qquad \text{if } W=0.
\end{array}
\tag{34}
$$

Eq. $(34)_4$ has the same form as (3); as conventional, we shall call "second quantization" the replacement $\psi \rightsquigarrow \varphi^H_\star$. If $\mathsf{H}^{(1)}_\star$ is t-independent, so is $\mathsf{H}_\star$, then $\mathsf{H}_\star(\varphi^H_\star) = \mathsf{H}_\star(\varphi)$, and (34) can be formulated directly in the Heisenberg picture as equations in the unknown $\varphi^H_\star(t)$. The map $\hat{\kappa}^H(t) = \hat{\kappa}\circ U(t) : \mathcal{H} \to \mathcal{X}_\star$ is a t-dependent $\widehat{U\mathbf{g}}$-equivariant unitary map, giving a t-dependent noncommutative configuration space realization of $\mathcal{H}$. We shall denote $\hat{\kappa}^H(e_i) = \hat{\varphi}_i(\hat{x})$.

Provided $\hat{V} = \wedge(V)$, $\hat{A}^a = \wedge(A^a)$ are well-defined, replacing $\hat{V}(\mathrm{x}\star,t) = V(\mathrm{x},t)\star$, $\hat{\mathrm{A}}(\mathrm{x}\star,t) = \mathrm{A}(\mathrm{x},t)\star$, $\hat{\varphi}_i(\mathrm{x}\star) = \varphi_i(\mathrm{x})\star$ we can reformulate the previous equations within $\hat{\Phi}^e$, $\hat{\Phi}$ using only $\star$-products, or equivalently, dropping $\star$-symbols and using only "hatted" objects:

$$
\begin{array}{ll}
\hat{\varphi}(\hat{\mathrm{x}}) = \hat{\varphi}_i(\hat{\mathrm{x}})\hat{a}'^i, & \varphi^{\hat{*}}(\hat{\mathrm{x}}) = \hat{a}^+_i\hat{\varphi}^{\hat{*}}_i(\hat{\mathrm{x}}) \\
[\hat{\varphi}(\hat{\mathrm{x}}), \hat{\varphi}(\hat{\mathrm{y}})]_\mp = \mathrm{h.c.} = 0, & [\hat{\varphi}(\hat{\mathrm{x}}), \hat{\varphi}^{\hat{*}}(\hat{\mathrm{y}})]_\mp = \hat{\varphi}_i(\hat{\mathrm{x}})\hat{\varphi}^{\hat{*}}_i(\hat{\mathrm{y}}), \\
i\hbar\frac{\partial}{\partial t}\hat{\psi}^{(n)} = \hat{\mathsf{H}}^{(n)}\hat{\psi}^{(n)}, & \hat{\mathsf{H}}^{(n)} = \sum\limits_{h=1}^{n}\hat{\mathsf{H}}^{(1)}(\hat{\mathrm{x}}_h, \hat{\partial}_h, t) + \sum\limits_{h<k}\hat{W}(\hat{\rho}_{hk}), \\
\multicolumn{2}{l}{\hat{\mathsf{H}} = \int_{\hat{X}}\!d\hat{\nu}(\hat{\mathrm{x}})\hat{\varphi}^{\hat{*}}(\hat{\mathrm{x}})\hat{\mathsf{H}}^{(1)}(\hat{\mathrm{x}},t)\hat{\varphi}(\hat{\mathrm{x}}) + \int_{\hat{X}}\!d\hat{\nu}(\hat{\mathrm{x}})\int_{\hat{X}}\!d\hat{\nu}(\hat{\mathrm{y}})\hat{\varphi}^{\hat{*}}(\hat{\mathrm{y}})\hat{\varphi}^{\hat{*}}(\hat{\mathrm{x}})W(\hat{\rho}_{\mathrm{xy}})\hat{\varphi}(\mathrm{x})\hat{\varphi}(\mathrm{y}),} \\
[\hat{\varphi}^H(\hat{\mathrm{x}},t), \hat{\varphi}^H(\hat{\mathrm{y}},t)]_\mp = \mathrm{h.c.} = 0, & [\hat{\varphi}^H(\hat{\mathrm{x}},t), \hat{\varphi}^{H\hat{*}}(\hat{\mathrm{y}},t)]_\mp = \hat{\varphi}_i(\hat{\mathrm{x}})\hat{\varphi}^{\hat{*}}_i(\hat{\mathrm{y}}), \\
i\hbar\frac{\partial}{\partial t}\hat{\varphi}_H = [\hat{\varphi}^H, \hat{\mathsf{H}}], \qquad \Rightarrow & i\hbar\frac{\partial}{\partial t}\hat{\varphi}^H = \hat{\mathsf{H}}^{(1)}\hat{\varphi}^H \qquad \text{if } W=0
\end{array}
\tag{35}
$$

Summing up: at least formally, we can formulate the same theory both on the commutative and on the noncommutative space, as summarized in (35). Solving the dynamics on one or the other will be a matter of convenience. In a minimalistic view, the noncommutative setting should be adopted only if the $\hat{\mathrm{x}}$-dependence of $\hat{V}(\hat{\mathrm{x}},t), \hat{\mathrm{A}}(\hat{\mathrm{x}},t), \hat{\varphi}_i(\hat{\mathrm{x}})$ is simpler than the x-dependence of $V(\mathrm{x},t)$, $\mathrm{A}(\mathrm{x},t)$, $\varphi_i(\mathrm{x})$, as it happens e.g. if the latter fulfill $\star$-differential equations.

In a non-conservative view, this construction suggests (35) as a general candidate framework for a **covariant nonrelativistic field quantization on the noncommutative spacetime $\mathbb{R}\times\hat{X}$ compatible with the axioms of quantum mechanics, including Bose/Fermi statistics**. Note in particular that in both the Schrödinger and Heisenberg picture the (anti)commutator of fields is a "c-number" distribution. The framework is not only $\widehat{U\mathbf{g}}$-, but also $\widehat{U\mathbf{g}'}$-covariant, where $\widehat{U\mathbf{g}'}$ is the Hopf $*$-algebra obtained from $U\mathbf{g}'$ by the same twist; to account for the t-dependence $C^1(\mathbb{R},\mathcal{H}), C^1(\mathbb{R},\mathcal{X})$,... must replace $\mathcal{H}, \mathcal{X}$,... as carrier spaces of the representations. Its consistency beyond the level of formal λ-power series has to be investigated case by case.

3.4 Nonrelativistic Quantum Mechanics on Moyal space $\mathbb{R}^3_\theta$: charged particle in a constant magnetic field

The definition (12)+(2)$_2$ for the Moyal deformation of the space of smooth functions of n copies of $\mathbb{R}^m$ (briefly $\mathbb{R}^m_\theta$) can be extended to larger domains in terms of Fourier transforms. For instance,

$$a(x_i)\star b(x_j)=\int d^mh\int d^mk\ e^{i\left(h\cdot x_i+k\cdot x_j-\frac{h\theta k}{2}\right)}\tilde{a}(h)\tilde{b}(k). \tag{36}$$

Here $\tilde{a}(k_i),\tilde{b}(k_j)$ are the Fourier transforms of $a(x_i),b(x_j)$, where $i,j=1,...,n$ and $x_i\equiv(x_i^h)$, $h=1,...,m$. The definition of $\wedge^n$ can be extended unambigously from the space of polynomials in x_i^h to a linear map $\wedge^n:\mathcal{X}'^{\otimes n}\to\widehat{\mathcal{X}'^{\otimes n}}$, simply by replacing $x_i\to\hat{x}_i$ in the Fourier decompositions:

$$\begin{aligned}&\wedge^n\left[\int d^mq_1...\int d^mq_n\ e^{iq_1\cdot x_1}...e^{iq_n\cdot x_n}\tilde{a}(q_1,...,q_n)\right]\\&:=\int d^mq_1...\int d^mq_n\ e^{iq_1\cdot\hat{x}_1}...e^{iq_n\cdot\hat{x}_n}e^{\frac{i}{2}\sum_{i<j}q_i\theta q_j}\tilde{a}(q_1,...,q_n).\end{aligned}\tag{37}$$

For $n=1$ (37) is nothing but the well-known Weyl transformation.
It is instructive to see explicitly how $\wedge^n$, acting on (anti)symmetric wavefunctions, "hides" their (anti)symmetry. Sticking to $n=2$, we find on the basis of plane waves

$$\wedge^2\left(e^{iq_1\cdot x_1}e^{iq_2\cdot x_2}\pm e^{iq_2\cdot x_1}e^{iq_1\cdot x_2}\right)=e^{iq_1\cdot\hat{x}_1}e^{iq_2\cdot\hat{x}_2}e^{\frac{i}{2}q_1\theta q_2}\pm e^{iq_2\cdot\hat{x}_1}e^{iq_1\cdot\hat{x}_2}e^{-\frac{i}{2}q_1\theta q_2}.$$

The (anti)symmetry remains manifest [15, 17] if we use coordinates ξ^a_i,X^a ($\xi^a_i:=x^a_{i+1}-x^a_i$, and $X^a:=\sum_{i=1}^n x^a_i/n$ are the coordinates of the center-of-mass of the system, which are completely symmetric). The map $\wedge^n$ deforms only the X part of the wavefunction, leaving unchanged and completely (anti)symmetric the ξ-part. The previous equation e.g. becomes

$$\wedge^2\left[e^{i(q_1+q_2)\cdot X}\left(e^{i(q_2-q_1)\cdot\xi_1}\pm e^{-i(q_2-q_1)\cdot\xi_1}\right)\right]=e^{i(q_1+q_2)\cdot\hat{X}}\left(e^{i(q_2-q_1)\cdot\hat{\xi}_1}\pm e^{-i(q_2-q_1)\cdot\hat{\xi}_1}\right)$$

As an example of a simple 1-particle model where the use of noncommutative coordinates helps solving the dynamics (3) we now consider a charged particle on $\mathbb{R}^3_\theta$ in a constant magnetic field B. The simplest gauge choice is $A^i(x)=\epsilon^{ijk}B^jx^k/2$. One finds that $\mathsf{H}^{(1)}_\star$ is still differential of second order, but more complicated than its undeformed (i.e. $\theta=0$) counterpart. In terms of "hatted" objects the model can be formulated and solved as in the undeformed case. We choose the x^3-axis parallel to $q\mathrm{B}=qB\vec{k}$ with $qB>0$, this gives $\hat{D}^3=\partial^3$, $\hat{D}^a=\partial^a-i\frac{qB}{2\hbar c}\epsilon^{ab}\hat{x}^b$ for $a,b\in\{1,2\}$, with $\epsilon^{12}=1=-\epsilon^{21}$, $\epsilon^{aa}=0$. These fulfill $[\partial^3,\hat{D}^a]=0$, $[\hat{D}^1,\hat{D}^2]=i\frac{qB}{\hbar c}[1-\frac{qB\theta^{12}}{2\hbar c}]$. Defining

$$a:=\alpha[\hat{D}^1-i\hat{D}^2],\qquad a^*=\alpha[-\hat{D}^1-i\hat{D}^2]\qquad \alpha:=\sqrt{\tfrac{\hbar c}{qB}}\Big/\sqrt{2-\tfrac{qB\theta^{12}}{2\hbar c}} \tag{38}$$

(we assume $qB\theta^{12}<4\hbar c$) one obtains the commutation relation $[a,a^*]=1$, and

$$\begin{aligned}&\hat{\mathsf{H}}^{(1)}=\tfrac{-\hbar^2}{2m}\hat{D}^i\hat{D}^i=\tfrac{-\hbar^2}{2m}\left[(\partial^3)^2-\tfrac{1}{2\alpha^2}(aa^*+a^*a)\right]=\hat{\mathsf{H}}^{(1)}{}_\parallel+\hat{\mathsf{H}}^{(1)}{}_\perp\\&\hat{\mathsf{H}}^{(1)}{}_\parallel:=\tfrac{(-i\hbar\partial^3)^2}{2m},\qquad \hat{\mathsf{H}}^{(1)}{}_\perp:=\hbar\omega\left(a^*a+\tfrac{1}{2}\right),\qquad \omega:=\tfrac{qB}{mc}\left(1-\tfrac{qB\theta^{12}}{4\hbar c}\right)\end{aligned}\tag{39}$$

It is easily checked that $[\hat{\mathsf{H}^{(1)}}_{\parallel}, \hat{\mathsf{H}^{(1)}}_{\perp}] = 0$. $\hat{\mathsf{H}^{(1)}}_{\parallel}$ has continuous spectrum $[0,\infty[$; the generalized eigenfuntions are the eigenfuntions $e^{ik\hat{x}^3}$ of $p^3 = -i\hbar\partial^3$ with eigenvalue $\hbar k$. The second is formally an harmonic oscillator Hamiltonian with ω modified by the presence of the noncommutativity θ^{12}. So the spectrum of $\hat{\mathsf{H}^{(1)}}$ is the set of $E_{n,k^3} = \hbar\omega(n+1/2) + (\hbar k^3)^2/2m$.
To find a basis of eigenfunctions we define in analogy with the undeformed case

$$\hat{z} := \sqrt{\tfrac{\zeta}{2}}(\hat{x}^1 + i\hat{x}^2), \quad \partial_{\hat{z}} := \tfrac{1}{\sqrt{2\zeta}}(\partial_1 - i\partial_2), \qquad \Rightarrow \qquad \hat{z}^* = \sqrt{\tfrac{\zeta}{2}}(\hat{x}^1 - i\hat{x}^2), \quad \partial_{\hat{z}}^* = -\partial_{\hat{z}^*}$$

with $\zeta := qB/2\hbar c$, and find that the only nontrivial commutators among $\hat{z}, \hat{z}^*, \partial_{\hat{z}}, \partial_{\hat{z}^*}$ are

$$[\partial_{\hat{z}}, \hat{z}] = 1, \qquad [\partial_{\hat{z}^*}, \hat{z}^*] = 1, \qquad [\hat{z}, \hat{z}^*] = \zeta\theta^{12}.$$

We can thus re-express a, a^* in the form

$$a = \alpha\sqrt{2\zeta}(\hat{z}^* + \partial_{\hat{z}}), \qquad a^* = \alpha\sqrt{2\zeta}(\hat{z} - \partial_{\hat{z}^*}).$$

Setting $\hat{l}^3 := \hat{z}\partial_{\hat{z}} - \hat{z}^*\partial_{\hat{z}^*} - \zeta\theta^{12}\partial_{\hat{z}}\partial_{\hat{z}^*}$ and $\mathrm{n} := a^* a$ we also find

$$[\hat{l}^3, \hat{z}^*] = -\hat{z}^*, \qquad [\hat{l}^3, \hat{z}] = \hat{z}, \qquad [\hat{l}^3, a^*] = a^*, \qquad [\hat{l}^3, a] = -a, \qquad [\hat{l}^3, \mathrm{n}] = 0. \tag{40}$$

In analogy with the undeformed case we can therefore choose as a complete set of commuting observables $\{p^3, \mathrm{n}, \hat{l}^3\}$. Let

$$\hat{\psi}_{0,0}(\hat{z}^*, \hat{z}) := \int dk\, dk^* e^{ik\hat{z}^*} e^{ik^*\hat{z}} e^{-kk^*} \qquad \Rightarrow \qquad a\hat{\psi}_{0,0} = 0 = \hat{l}^3\hat{\psi}_{0,0}. \tag{41}$$

(when $\theta = 0$ this gives $\psi_{0,0} \propto e^{-zz^*}$). The *deformed Landau eigenfunctions*

$$\hat{\psi}_{k^3,n,m}(\hat{\mathrm{x}}) = (a^*)^n (\hat{z}^*)^{n-m} \hat{\psi}_{0,0}(\hat{z}^*, \hat{z}) e^{ik^3\hat{x}^3} \tag{42}$$

are generalized eigenfunctions with eigenvalues $p^3 = \hbar k^3 \in \mathbb{R}$, $, \mathrm{n} = n = 0, 1, ...,$ $\hat{l}^3 = m = n, n-1, ...$ and build up an orthogonal basis of $\mathcal{L}^2(\mathbb{R}^3)$. They are also eigenfunctions of $\hat{\mathsf{H}^{(1)}}$ with eigenvalues $E_{n,k^3} = \hbar\omega(n+1/2) + (\hbar k^3)^2/2m$. Each energy level, in particular the lowest one, has ∞-ly many different eigenfunctions, which are characterized by the same n, k^3 and different m's. Replacing $\hat{x}^a \to x^a\star$ and performing all the $\star$-products one finds the corresponding eigenfuntions $\psi_{k^3,n,m}(\mathrm{x})$ of $\mathsf{H}^{(1)}_\star$. Their x-dependence is messy, whereas the $\hat{\mathrm{x}}$-dependence of the (42) is simple and practically the same as in the undeformed counterpart, as anticipated.

4 Relativistic second quantization

By analogous considerations one can construct (at least) a consistent free QFT on a noncommutative Minkowski spacetime with twisted symmetry. The commutative manifold X is Minkowski spacetime with coordinates $x \equiv (x^\mu)$ ($\mu = 0, 1, 2, 3$ and $x^0 \equiv t$) w.r.t. a fixed inertial frame, and G is its symmetry group, the Poincaré Lie group. A relativistic particle is described choosing as the algebra of observables $\mathcal{O} = H = U\mathbf{g}$ and as the Hilbert space $\overline{\mathcal{H}}$ the completion of a pre-Hilbert space $\mathcal{H}$ carrying an irreducible $*$-representation of

$U\mathbf{g}$ charaterized by a nonnegative eigenvalue m^2 of the Casimir $P^\mu P_\mu$ and a nonnegative spectrum for P^0. We stick to the case of a scalar particle of positive mass m.
The relevant space of functions is $\mathcal{X}:=\kappa^H(\mathcal{H})\subset\mathcal{S}(\mathbb{R}^4)$, the pre-Hilbert space of rapidly decreasing, smooth, positive-energy solutions of the Klein-Gordon equation. The map κ^H is the usual $U\mathbf{g}$-equivariant (t-dependent) commutative configuration space realization of $\mathcal{H}$. We denote $\kappa^H(e_i)=\varphi_i$ (these functions depend both on space and time) and $\Phi^e=\mathcal{A}^{\pm}\otimes(\bigotimes_{i=1}^{\infty}\mathcal{X}')$, where $\mathcal{X}'$ stands for the dual space of $\mathcal{X}$. Fixed a twist $\mathcal{F}\in(H\otimes H)[[\lambda]]$ (e.g. the Moyal twist, or the twists classified in [6]) we apply the associated $\star$-deformation procedure $H\to\hat{H}, \Phi^e\to\Phi^e_\star$, etc. The hermitean relativistic free field (in the Heisenberg picture) is expressed in the form

$$\varphi(x)=\varphi_i(x)\star a'^i+a_i^+\star\varphi_i^{*\star}(x)=\int\frac{d^3p}{2p^0}[e^{-ip\cdot x}\star a'^{\mathrm{p}}+a^+_{\mathrm{p}}\star e^{ip\cdot x}] \tag{43}$$

and the free field commutation relation in the form

$$[\varphi(x)\overset{\star}{,}\varphi(y)]=\varphi_i(x)\star\varphi_i^{*\star}(y)-\varphi_i(y)\star\varphi_i^{*\star}(x)=\int\frac{d^3p}{2p^0}\left[e^{-ip(x-y)}-e^{ip(x-y)}\right] \tag{44}$$

for any $x,y\in\{x_1,x_2,...\}$. This vanishes if $x-y$ is space-like (microcausality). We have expressed the right-hand sides both in terms of a generic normalizable basis $\kappa^H(e_i)=\varphi_i$ of $\mathcal{X}$ and in terms of the generalized basis $\kappa^H(e_{\mathrm{p}})=e^{-ip\cdot x}$ of eigenvectors of the momentum operators $\tilde{\kappa}^H(P_\mu)=i\partial_\mu$ with eigenvalues p_μ $[(p^a)\equiv\mathrm{p}\in\mathbb{R}^3$ and $p^0\equiv\sqrt{\mathrm{p}^2+m^2}>0]$; $a^{\mathrm{p}}, a^+_{\mathrm{p}}$ are the associated generalized creation & annihilation operators, which fulfill

$$[a^{\mathrm{p}},a^{\mathrm{q}}]=0,\qquad [a^+_{\mathrm{p}},a^+_{\mathrm{q}}]=0,\qquad [a^{\mathrm{p}},a^+_{\mathrm{q}}]=2p^0\delta^3(\mathrm{p}-\mathrm{q}).$$

Assuming that $\mathcal{F}$ is such that $\wedge$ is well-defined on the whole of $\mathcal{X}$ and going to the "hat notation" we find

$$\begin{aligned}&\hat{\varphi}(\hat{x})=\hat{\varphi}_i(\hat{x})\hat{a}'^i+\hat{a}_i^+\hat{\varphi}_i^{\hat{*}}(\hat{x})\\&[\hat{\varphi}(\hat{x}),\hat{\varphi}(\hat{y})]=\hat{\varphi}_i(\hat{x})\hat{\varphi}_i^{\hat{*}}(\hat{y})-\hat{\varphi}_i(\hat{y})\hat{\varphi}_i^{\hat{*}}(\hat{x})\\&(\hat{\Box}+m^2)\hat{\varphi}(\hat{x})=0\end{aligned} \tag{45}$$

Choosing the Moyal twist $(2)_2$ one obtains Moyal-Minkowski noncommutative spacetime and the twisted Poincaré Hopf algebra of [7, 22, 18]. In terms of generalized creation & annihilation operators formulae (21), (22) and the states created by the deformed creation operators become

$$\begin{aligned}&a^+_{\mathrm{p}}\star a^+_{\mathrm{q}}=e^{-ip\theta q}\,a^+_{\mathrm{q}}\star a^+_{\mathrm{p}},\\&a^{\mathrm{p}}\star a^{\mathrm{q}}=e^{-ip\theta q}\,a^{\mathrm{q}}\star a^{\mathrm{p}},\\&a^{\mathrm{p}}\star a^+_{\mathrm{q}}=e^{ip\theta q}\,a^+_{\mathrm{q}}\star a^{\mathrm{p}}+2p^0\delta^3(\mathrm{p}-\mathrm{q})\\&a^{\mathrm{p}}\star e^{iq\cdot x}=e^{-ip\theta q}\,e^{iq\cdot x}\star a^{\mathrm{p}},\quad\&\text{ h.c.,}\end{aligned}\quad\Leftrightarrow\quad\begin{aligned}&\hat{a}^+_{\mathrm{p}}\hat{a}^+_{\mathrm{q}}=e^{iq\theta p}\,\hat{a}^+_{\mathrm{q}}\hat{a}^+_{\mathrm{p}},\\&\hat{a}^{\mathrm{p}}\hat{a}^{\mathrm{q}}=e^{iq\theta p}\hat{a}^{\mathrm{q}}\hat{a}^{\mathrm{p}},\\&\hat{a}^{\mathrm{p}}\hat{a}^+_{\mathrm{q}}=e^{ip\theta q}\hat{a}^+_{\mathrm{q}}\hat{a}^{\mathrm{p}}+2p^0\delta^3(\mathrm{p}-\mathrm{q}),\\&\hat{a}^{\mathrm{p}}e^{iq\cdot\hat{x}}=e^{-ip\theta q}\,e^{iq\cdot\hat{x}}\hat{a}^{\mathrm{p}},\quad\&\text{ h.c.;}\end{aligned} \tag{46}$$

$$\check{a}^+_{\mathrm{p}}\equiv D^\sigma_{\mathcal{F}}\left(a^+_{\mathrm{p}}\right)=a^+_{\mathrm{p}}e^{-\frac{i}{2}p\theta\sigma(P)},\qquad \check{a}^{\mathrm{p}}\equiv D^\sigma_{\mathcal{F}}\left(a^{\mathrm{p}}\right)=a^{\mathrm{p}}e^{\frac{i}{2}p\theta\sigma(P)} \tag{47}$$

$$\hat{a}^+_{\mathrm{p}_1}...\hat{a}^+_{\mathrm{p}_n}|0\rangle=a^+_{\mathrm{p}_1}\star...\star a^+_{\mathrm{p}_n}|0\rangle=\check{a}^+_{\mathrm{p}_1}...\check{a}^+_{\mathrm{p}_n}|0\rangle=\exp\left[-\frac{i}{2}\sum_{\substack{j,k=1\\j<k}}^{n}p_j\theta p_k\right]a^+_{\mathrm{p}_1}...a^+_{\mathrm{p}_n}|0\rangle \tag{48}$$

where the Jordan-Schwinger map takes the form $\sigma(P_\mu) = \int d\mu(p)\, p_\mu a^+_{\rm p} a^{\rm p}$. By (48) generalized states differ from their undeformed counterparts only by multiplication by a phase factor. As $\check{a}^+_{\rm p}\check{a}^{\rm p} = a^+_{\rm p} a^{\rm p}$, $\sigma(P_\mu) = \int d\mu(p)\, p_\mu \check{a}^+_{\rm p}\check{a}^{\rm p}$, from (47) the inverse of $D^\sigma_{\mathcal{F}}$ is readily obtained.
It is remarkable that the free field (43) & (46) coincides with the one found in formulae (37) & (46) of [15] [see also formulae (32) & (36) of [16]] imposing just the free field equation and Wightman axioms (modified only by the requirement of *twisted* Poincaré covariance). See [15, 16, 17] for comparisons with other approaches considered in the literature. In [15] it has been also shown that the n-point functions of a (at least scalar) field theory, when expressed as functions of coordinates differences $\xi^\mu_i := x^\mu_{i+1} - x^\mu_i$, coincide with the undeformed ones. Quite disappointingly, this result holds in time-ordered perturbation theory also for interacting fields with interaction $\varphi^{\star n}$, due to the translation invariance of the latter, but should no more hold for e.g. a quantum matter field interacting with a gauge field.

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